Quantum Fields in Non-Static Background: A Histories Perspective

C. Anastopoulos *

Departament de Fisica Fonamental, Universitat de Barcelona

Av. Diagonal 647, 08028 Barcelona, Spain

and

Department of Physics, University of Maryland, College Park, MD20742, USA

(current address)

February 4, 2008

Abstract

For a quantum field living on a non - static spacetime no instantaneous Hamiltonian is definable, for this generically necessitates a choice of inequivalent representation of the canonical commutation relations at each instant of time. This fact suggests a description in terms of time - dependent Hilbert spaces, a concept that fits naturally in a (consistent) histories framework. Our primary tool for the construction of the quantum theory in a continuous -time histories format is the recently developed formalism based on the notion of the history group . This we employ to study a model system involving a 1+1 scalar field in a cavity with moving boundaries. The instantaneous (smeared) Hamiltonian and a decoherence functional are then rigorously defined so that finite values for the time - averaged particle creation rate are obtainable through the study of energy histories. We also construct the Schwinger - Keldysh closed- time - path generating functional as a "Fourier transform" of the decoherence functional and evaluate the corresponding n - point functions.

---

*E-mail:charis@physics.umd.edu

1

I   Introduction

The consistent histories approach [1, 2, 3, 4] was mainly devised as an alternative point of view to quantum phenomena, providing a more convenient language for the treatment of individual, closed quantum mechanical systems. While its physical predictions exactly agree with the ones of standard quantum mechanics (arguably even for the case of the paradoxes connected to the multiplicity of the consistent sets), its internal structure is somehow distinct. While standard quantum mechanics (in its Heisenberg version) incorporates kinematics through Hilbert space operators, dynamics through a Hamiltonian and initial conditions (probability assignment) through a state (density matrix), in the history theory histories (or history propositions) provide the kinematics , with state and dynamics being encoded in a new object: the decoherence functional.

This is a complex valued function $d(\alpha, \alpha')$ of pairs of histories, whose role is the assignement of probabilities. If for any set of histories, the decoherence condition

$$d(\alpha, \alpha') = 0 \qquad\qquad (\text{I. 1})$$

is satisfied for $\alpha \neq \alpha'$, then a probability measure exists in this set given by $p(\alpha) = d(\alpha, \alpha)$.

A mathematically elegant formalism for histories has been developed by Isham and collaborators [5, 6] . In this formulation histories can be identified with projection operators on a Hilbert space. In standard systems this is constructed from the tensor product of the single - time Hilbert spaces, that characterise the canonical theory. Besides providing a characterisation of consistent histories as temporal quantum logic, this formalism highlights the its similarity to their closest classical analogue: stochastic processes. An important feature of this construction is that these single-time Hilbert spaces need not be isomorphic (or carry isomorphic structures such as unitarily equivalent group representations) to each other.

This seems particularly suited for the study of quantum field theories in non-static background, for the following reason : The Hilbert space that defines the quantum theory corresponding to a particular classical system is constructed from the study of the representations of the canonical group (for the general scheme see reference [7]). For linear systems this is the familiar Weyl group, the Lie algebra of which is defined by the canonical commutation relations. When considering fields we have to deal with an infinite dimensional Lie group, which will admit many unitarily inequivalent representations. The natural way to proceed would be to select a representation, in which the Hamiltonian can defined as a concrete self -adjoint operator. When this is attempted for fields in a non-static background, one realises that at different times one has to admit unitarily inequivalent representations of the canonical commutation relations. This implies the non -existence of an instantaneous Hamiltonian. Rather than abandoning the definition of a Hamiltonian, the histories approach provides a

possibility of welding these representations together in order to construct a well-defined *finite* quantum theory describing such systems.

Two recent developements provide insight necessary for dealing with this case. In a series of papers Isham *et al* analysed the kinematical structure of the Hilbert space describing continuous - time histories [8, 9]. The main ingredient has been the history group, the analogue of the canonical group in the histories context. Its Lie algebra for the case of a particle at a line is

$$[x_t, p_{t'}] = i\hbar\delta(t, t') \tag{I. 2}$$

The time index t does not here refer to the dynamics of the system, as generated by a Hamiltonian, but is an index labeling the instant of time at which a proposition (for instance corresponding to the generators) is asserted. Equation (1.2) is formally similar to the canonical algebra for an 1+1 field theory and as such admits many unitarily inequivalent representations. A guiding principle for a selection of a representation has been the definability of an instantaneous Hamiltonian [9] . Accepting continuous time implies that all history propositions are about quantities smeared (averaged) in time. This again suggests that if we demand the existence of a smeared, instantaneous Hamiltonian, we might be able to obtain a unique representation of the history algebra corresponding to a quantum field in non-static background.

This we shall show that can be relatively straightforwardly achieved through a simple generalisation of the results of reference [9]. But then we should need a guiding principle for the construction of the decoherence functional, since the corresponding canonical theory is not well defined . This has come from a recent result by Savvidou: the discernment of two laws of time transformations (and corresponding time parameters) in the backbone of the structure of history theories [10]. One parameter is associated with the background temporal structure and describes how one moves from one single - time Hilbert space to another (Schrödinger time). The other incorporates the effects of the actual dynamics (Heisenberg time). Taking this as a fundamental property that ought to be reflected in all objects of our theory , we have been able to expand on a previous partial result [8] and identify three pieces out of which a physical decoherence functional is constructed. These pieces we call: the Schrödinger operator, the Heisenberg operator and the boundary operator. They correspond respectively to the aforementioned times and the initial state.

This result enables us then to write a *finite* decoherence functional for a model case we study in this paper: an 1+1 field in a cavity with moving boundaries. This is sufficiently general to prove the main point: that a history theory based on the physical principle of the existence of an instantaneous Hamiltonian rigorously exists and can provide finite values for the probabilities assigned to histories [1]. In addition, we establish that the Schwinger - Keldysh closed - time

---

[1]One can view these results as a direct consequence of the existence of two notions of time transformation in history theory compared to the unique one of canonical quantum mechanics.

- path (CTP) generating functional [11] is equal to the decoherence functional evaluated at a pairs of elements of the history group. Hence we are able to construct n - point functions and get into contact with the results of the more familiar canonical treatment.

The generalisation of these results to general spacetimes is technically straightforward (by demanding the existence of the Ashtekar-Magnon Hamiltonian [12]), but for one thing. Our theory should be independent of the choice of the time variable employed in the definition of the Hamiltonian. This means that changes of foliation should be generically implemented by a unitary operator on the history Hilbert space [13]. Such a proof of unitary implementation for the general case is quite more demanding and necessitates a different set of techniques from the ones we employ in this paper. We therefore defer it to a future work. Here we shall restrict ourselves to the convenient choice of the background Minkowski time, which in any case is relevant for the discussion of the time - dependent Casimir effect.

II   Structure of decoherence functional

All physical histories can be represented by elements of a lattice of propositions [5] and in the familiar case of standard quantum mechanics they are realised by projection operators on a Hilbert space $\mathcal{F}$ which is the tensor product of the single time Hilbert spaces

$$\mathcal{F} = H_{t_1} \otimes H_{t_2} \otimes \ldots \otimes H_{t_n} \qquad \text{(II. 1)}$$

In [6] the most general form of a decoherence functional satisfying the relevant axioms has been constructed. It is in one to one correspondence with particular class of operators X that act on $\mathcal{F} \otimes \mathcal{F}$. Explicitly

$$d(\alpha, \alpha') = Tr_{\mathcal{F} \otimes \mathcal{F}} \left(P_\alpha \otimes P_{\alpha'} X \right) \qquad \text{(II. 2)}$$

This is an important result, enabling a mathematical classification of decoherence functionals, but what would be more interesting for physical aplications is the construction of the decoherence functional in terms of operators acting solely on $\mathcal{F}$. (This has been done for a special case in reference [8]). The reason for that is the possibility of having a physical interpretation and understanding for such objects. This would enable the construction of the decoherence functional even when not having the reliable guide of a corresponding canonical theory. This is what we shall undertake in this section.

In the canonical theory the non - homogeneity of time transformations reflects itself in the non - existence of the corresponding generator. In the history version it is only the generator of *kinematical* time transformations, related to the spacetime causal structure (the Liouville operator of reference [10]) that need not exist, while the other (the Hamiltonian) does exist, and it is the one that determines the representation space of the theory.

Our starting point is the standard form of the time symmetric decoherence functional [4]

$$d(\alpha, \alpha') = Tr_H(C_{\alpha'}^\dagger \rho_f C_\alpha \rho_0)/Tr(\rho_f \rho_0) \tag{II. 3}$$

with $C_\alpha = P_{\alpha_n}(t_n) \dots P_{\alpha_1}(t_1)$ in terms of the Heisenberg picture projectors and the trace is performed within the single-time Hilbert space H. Recall that the standard form can be obtained by setting $\rho_f = 1$.

It is easy to verify [6] that the above expression can be written in the form

$$Tr_{\otimes_{i=1}^4 H_i} \left(C_{\alpha'}^\dagger \otimes \rho_f \otimes C_\alpha \otimes \rho_0 S_4 \right) / Tr(\rho_f \rho_0) \tag{II. 4}$$

where S_4 is an operator acting on $H \otimes H \otimes H \otimes H$ as

$$S_4(|v_1\rangle \otimes |v_2\rangle \otimes |v_3\rangle \otimes |v_4\rangle) = |v_2\rangle \otimes |v_3\rangle \otimes |v_4\rangle \otimes |v_1\rangle \tag{II. 5}$$

Tracing independently over the second and fourth Hilbert space in (2.4) we get the expression $Tr_{H \otimes H}(C_{\alpha'}^\dagger \otimes C_\alpha Z)$ where

$$Z = Tr_{H_2 \otimes H_4}(1 \otimes \rho_f \otimes 1 \otimes \rho_0)S_4/Tr(\rho_f \rho_i) \tag{II. 6}$$

in an obvious notation. The matrix elements of Z in an orthonormal basis $|i\rangle \otimes |j\rangle$ of $H \otimes H$ are easily computed

$$\langle ml|Z|ij\rangle = (\rho_f)_{mj}(\rho_0)_{ki}/Tr(\rho_f \rho_0) \tag{II. 7}$$

This can be written in the form

$$\langle ml|Z|ij\rangle = \sum_{rs} \left[(\rho_f^{1/2})_{mr}(\rho_0^{1/2})_{si}\right] \left[(\rho_f^{1/2})_{rj}(\rho_0^{1/2})_{ls}\right] /Tr(\rho_f \rho_0) \tag{II. 8}$$

hence

$$Z = \sum_{rs} A^{(r,s)} \otimes (A^\dagger)^{(r,s)} \tag{II. 9}$$

with $A^{(rs)}$ operators on H with matrix elements

$$\langle i|A^{(rs)}|j\rangle = \left[(\rho_f^{1/2})_{ir}(\rho_0^{1/2})_{sj}\right] / (Tr(\rho_f \rho_o))^{1/2} \tag{II. 10}$$

Since the history operators are trace-class and the A's bounded, the decoherence functional can be written

$$d(\alpha, \alpha') = \sum_{rs} Tr_H \left(C_{\alpha'}^\dagger A^{(rs)} \right) Tr_H \left(C_\alpha A^{\dagger(rs)} \right) \tag{II. 11}$$

In the above expression the decoherence functional has separated in different traces the contribution of each of the pair of histories. If in each of those traces we employ the technique we used to derive (2.4) we obtain

$$d(\alpha, \alpha') = \sum_{rs} Tr_{\mathcal{F}} \left[\mathcal{U}^\dagger P_\alpha \mathcal{U} \mathcal{A}^{\dagger(rs)} \mathcal{R} \mathcal{S} \mathcal{R} \right] Tr_{\mathcal{F}} \left[\mathcal{U}^\dagger P_{\alpha'} \mathcal{U} \mathcal{A}^{(rs)} \mathcal{S} \right] \tag{II. 12}$$

5

where

$$P_\alpha = P_{\alpha_1} \otimes \ldots \otimes P_{\alpha_n} \tag{II. 13}$$

is the projector on $\mathcal{F}$ corresponding to the history proposition α and similarly for $P_{\alpha'}$. Also

$$\mathcal{U} = U_1 \otimes \ldots U_n = e^{-iHt_1} \otimes \ldots \otimes e^{-iHt_n} \tag{II. 14}$$

$$\mathcal{A}^{(rs)} = A^{(rs)} \otimes 1 \otimes \ldots \otimes 1 \tag{II. 15}$$

$$\tag{II. 16}$$

and the operators $\mathcal{S}$ and $\mathcal{R}$ are defined in terms of their action

$$\mathcal{S}(|v_1\rangle \otimes \ldots \otimes |v_n\rangle) = |v_2\rangle \otimes |v_3\rangle \otimes \ldots \otimes |v_1\rangle \tag{II. 17}$$

$$\mathcal{R}(|v_1\rangle \otimes \ldots \otimes |v_n\rangle) = |v_n\rangle \otimes |v_{n-1}\rangle \otimes \ldots \otimes |v_1\rangle \tag{II. 18}$$

Before discussing the physical significance of these operators let us recast our expressions in a more elegant and suggestive form. Let us by $\partial_-\mathcal{F}$ and $\partial_+\mathcal{F}$ denote the past and future "boundary" of $\mathcal{F}$, that is the Hilbert spaces H_{t_1} and H_{t_n} in the case of discrete histories we have considered in this section. If $\mathcal{H}$ is the space of continuous linear maps $\partial_-\mathcal{F} \to \partial_+\mathcal{F}$ then $\mathcal{A}$ can be viewed as a linear map from $\mathcal{F}$ to $\mathcal{H}$. Checking that $\mathcal{R}\mathcal{S}\mathcal{R} = \mathcal{S}^\dagger$ and assuming the initial and final times to be the ones at which ρ_0 and ρ_f are defined our expression (2.12) reads

$$d(\alpha, \alpha') = Tr_{\mathcal{H}} \left[Tr_{\mathcal{F}} \left(\mathcal{U}^\dagger P_\alpha \mathcal{U}(\mathcal{S}\mathcal{A})^\dagger \right) Tr_{\mathcal{F}} \left(\mathcal{U}^\dagger P_{\alpha'} \mathcal{U}(\mathcal{S}\mathcal{A}) \right) \right] \tag{II. 19}$$

hence the operator X of (2.2) is given by

$$X = Tr_{\mathcal{H}} \left(\mathcal{U}\mathcal{A}^\dagger \mathcal{S}^\dagger \mathcal{U}^\dagger \otimes \mathcal{U}\mathcal{S}\mathcal{A}\mathcal{U}^\dagger \right) \tag{II. 20}$$

II.1 Interpretation

It is clear from the discussion above that three are the important ingredients entering in the construction of the decoherence functional:

1. The unitary operator $\mathcal{U}$ in which the contribution of the dynamics in the evaluation of probabilities are contained. Its action is to provide the weight in the probabilities due to time evolution, or, rather heuristically, to turn the projection operators into Heisenberg picture ones. Note that this operator can be similarly defined even when the Hamiltonian is time dependent.

2. The operator $\mathcal{S}$. As can be seen from its definition it encodes the temporal structure (in the sense of partial ordering) of the history theory. In the standard (discrete time) case we have examined, it can be readily verified to be

unitary, but it seems reasonable that this condition could be relaxed in certain generalisations [2]. A sufficient condition for the finiteness of the traces in (2.19) is that $\mathcal{S}$ is bounded, which follows trivially when being unitary. Note that also by its definition $Tr\mathcal{S} = 1$.

3. The linear map $\mathcal{A}$. It essentially contains the contribution of the initial and final states, that is the weight given to probabilities by the particulary boundary conditions. In the standard case it is a trace-class operator but in general (as in our particular examples later) we can dispense with even its (strong)continuity. Given our previous assumptions for $\mathcal{U}$ and $\mathcal{S}$ a sufficient condition on $\mathcal{A}$ for the finiteness of the traces is the

Weak continuity condition:
1. $Tr_{\mathcal{H}}\left(\mathcal{A}^{\dagger}\mathcal{A}\right) : \mathcal{F} \otimes \bar{\mathcal{F}} \otimes \mathcal{F} \otimes \bar{\mathcal{F}} \to \mathbf{C}$ is continuous.
2. For any $|\phi_1\rangle, |\phi_2\rangle \in \mathcal{H}$, $\langle\phi_1|\mathcal{A}|\phi_2\rangle : \mathcal{F} \to \mathbf{C}$ is continuous.

For purposes of easy reference we shall henceforward call $\mathcal{U}$ the Heisenberg operator, $\mathcal{S}$ the Schrödinger operator and $\mathcal{A}$ the boundary operator.

Casting the decoherence functional in the form (2.19) can be seen as a step of departure for constructing generalised history theories. A non-trivial generalisation is when the single-time Hilbert spaces are not the same . The change is then included in the operator $\mathcal{S}$ which contains information about the welding of the unequal time Hilbert spaces together in the history Hilbert space $\mathcal{F}$. If there exist (physically justifiable) identification maps between the Hilbert spaces H_{t_i} (to keep full generality this does not have to be structure preserving), i.e. a family of maps

$$I(t_i, t_j) : H_{t_j} \to H_{t_i} \tag{II. 21}$$

then the Schrödinger operator can be defined as

$$\mathcal{S}(|v_1\rangle \otimes |v_2\rangle \otimes \ldots \otimes |v_n\rangle) = I(t_2, t_1)|v_1\rangle \otimes I(t_3, t_2)|v_2\rangle \otimes \ldots I(t_1, t_n)|v_n\rangle \tag{II. 22}$$

Finally, we should remark that the behaviour of $\mathcal{S}$ at the boundaries would results in most situations in its intricate mixing with the boundary operator $\mathcal{A}$.

III   Scalar field in time dependent spacetime

The identification of the operator structure within the decoherence functional carried out in the previous section, enables us to proceed in the construction of history theories, in which there is a time-dependence on the single time Hilbert spaces. Our motivation is the study of field theories in non - static background; hence in the rest of the paper we shall undertake an examination of a simple model, employing techniques developed for the study of continuous histories.

---

[2] What we have in mind is spacetimes that are not globally hyperbolic, as possibly involving topology change.

III.1 The history algebra

The system under study is a massless scalar field in a cavity with time-varying size given by the positive function $L(t)$ (this has to be assumed to have at least continuous first derivative). On the edges of the cavity Dirichlet boundary conditions are to be assumed. The time parameter with respect to which the history theory is defined is the "Minkowski" time.

Our natural choice for the history algebra is

$$[\phi_t(x), \pi_{t'}(x')] = i\frac{1}{L(t)}\delta(t,t')\delta_I(x,x') \tag{III. 1}$$

where t, t' lie in $\mathbf{R}$, x, x' in $I = [0, 1]$, δ_I denotes the delta function as defined on I and Dirichlet boundary conditions are assumed for the fields [3]. More precisely one should use the smeared fields

$$\phi(f) = \int dt L(t) \int_0^1 dx \phi_t(x) f(t,x) \tag{III. 2}$$

$$\pi(g) = \int dt L(t) \int_0^1 dx \phi_t(x) g(t,x) \tag{III. 3}$$

where f and g are elements of the vector space $L^2_R(R) \otimes L^2_R(I)_D$ where D stands for the imposition of the Dirichlet boundary condition

$$f(0,t) = f(1,t) = 0 \tag{III. 4}$$

This way we have

$$[\phi(f), \pi(g)] = i \int dt L(t) \int_0^1 dx f(t,x) g(t,x) \tag{III. 5}$$

It is more convenient to express the fields in terms of their Fourier transforms

$$\phi_t(x) = L^{-1/2}(t) \sum_{n=1}^{\infty} q_t(n) \sin n\pi x \tag{III. 6}$$

$$\pi_t(x) = L^{-1/2}(t) \sum_{n=1}^{\infty} p_t(n) \sin n\pi x \tag{III. 7}$$

with respect to which

$$[q_t(n), p_{t'}(m)] = i\delta_{nm}\delta(t,t') \tag{III. 8}$$

These can be smeared by elements of $L^2_R(R)_D$, so that $q_n(f) = \int dt q_t(n) f(t)$. Hence

$$[q_n(f), p_m(g)] = i\delta_{mn} \int dt f(t) g(t) \tag{III. 9}$$

and the underlying vector space of the history algebra is simply $L^2_R(R)_D \otimes l^2_R$.

---

[3] The δ function is represented as $\sum_{n=1}^{\infty} \sin n\pi x$. The presence of the $1/L(t)$ in the right hand side, is due to the fact that the proper integration measure is $L(t)dx$.

III.2 The representation space

Our next task is to find a representation of the history algebra (3.1) in a Hilbert space $\mathcal{F}$, together with an isometry from the continuous tensor product $\otimes_{t\in R}\mathcal{H}_t$, where $\mathcal{H}_t$ is the Hilbert space on which the *canonical algebra* at time t is represented. This Hilbert space supposedly contains the projections having information about the properties of of the system at time t.

III.2.1 Single-time Hilbert spaces $\mathcal{H}_t$

The canonical algebra at time t reads

$$[\phi(x), \pi(x')] = \frac{i}{L(t)}\delta_I(x - x') \qquad\qquad \text{(III. 10)}$$

Now $\mathcal{H}_t$ has naturally the structure of an exponential Hilbert space: $\mathcal{H}_t = e^{V_t} = \oplus_{n=0}^{\infty}(V_t)_S^n$ where V_t is the space of complex valued functions on $[0,1]$ satisfying Dirichlet boundary conditions and with inner product given by

$$(z_1, z_2)_t = L(t)\int_0^1 dx z_1^*(x)z_2(x) \qquad\qquad \text{(III. 11)}$$

It is well known that $\mathcal{H}_t$ is spanned by an overcomplete set of states (unnormalised coherent states) $|\exp z\rangle = \oplus_{n=0}^{\infty} \otimes_n z,\ (z \in V_t)$ with inner product

$$\langle\exp z| \exp w\rangle_t = \exp\left(L(t)\int_0^1 dx z^*(x)w(x)\right) \qquad\qquad \text{(III. 12)}$$

Equivalently using the Fourier transform

$$f(x) = L^{-1/2}\sum_{n=1}^{\infty} z_n \sin n\pi x \qquad\qquad \text{(III. 13)}$$

one characterises V_t as $L^2(R)_D \otimes l^2$.

III.2.2 The history Hilbert space $\mathcal{F}$

The fact that $\mathcal{H}_t$ can be writen as an exponential Hilbert space enables us to employ the analysis of [8] for the construction of $\mathcal{F}$. The important identity that carries on into our case is

$$\langle\otimes_t \exp z_t| \otimes_t \exp w_t\rangle_{\otimes_t e^{V_t}} = \exp\left(\int dt(z_t|w_t)_{V_t}\right)$$

$$= \exp\left(\int dt \sum_{n=1}^{\infty}(z_n^*)_t(w_n)_t\right) \qquad\qquad \text{(III. 14)}$$

9

where z and w stand for elements of l^2. This implies straightforwardly the isomorphisms

$$\otimes_t \exp V_t \quad \simeq \quad \exp \int^{\oplus} V_t \simeq \exp \mathcal{E}$$

$$\otimes_t |\exp z_t\rangle \quad \rightarrow \quad |\exp \int^{\oplus} z_t dt\rangle \rightarrow |\exp z(.)\rangle \qquad \text{(III. 15)}$$

where $\mathcal{E}$ is the space $L^2(R) \otimes l^2$ (i.e. some complexification of the test function space in the history algebra) with inner product

$$\langle z|w\rangle_{\mathcal{E}} = \int dt \sum_{n=1}^{\infty} z_n^*(t) w_n(t) \qquad \text{(III. 16)}$$

So we conclude that

$$\mathcal{F} = \otimes_t \mathcal{H}_t \simeq \exp \mathcal{E} \qquad \text{(III. 17)}$$

III.3   Operators on F

Having found the structure of $\mathcal{F}$ does not yet mean that we have identified the representation of the canonical group. In each Fock space construction, creation and annihilation operators are naturally defined (see for instance [14]), but we have still to determine how the generators of the history group are written as linear combinations of them (equivalent to choosing a complex structure on $L_R^2(R) \otimes l_R^2$, the space of smearing functions of the history group [15]).

III.3.1   The Hamiltonian operator

In standard quantum field theory in Minkowski spacetime this is achieved by postulating that the state corresponding to the vector $|\exp 0\rangle$ is invariant under the action of the Poincaré group as is represented on the Fock space [16], or equivalently that it is the lowest eigenvalue of the field 's Hamiltonian. In generalising to field theory in curved spacetime one demands that this state is invariant under the spacetime group of isometries. In general, it is essential that this group has a generator that corresponds to a timelike Killing vector field , in order that time evolution can be unitarily implemented.

   This interconnection between the choice of the representation and the existence of a Hamiltonian type of operator has been exploited for the case of the history group in [9], in order both to select a Fock space representation among the ones considered in [8] and construct an operator, the spectral family of which can naturally be said to correspond to history propositions about energy.

   In our system, the absence of a time translation symmetry would disallow the definition of a Hamiltonian in a canonical framework (mainly for the inability to select a representation of the CCR), but there is nothing forbidding

the introduction of a smeared Hamiltonian in the corresponding history theory, along the lines of [9]. This would correspond to the spatially integrated 00 component of the energy momentum tensor and should act as an operator governing the evolution within a single - time Hilbert space.

Keeping these remarks in mind, we can now proceed to the introduction of the Hamiltonian H_t. Formally, this should be

$$H_t = L(t) \int_0^1 dx \frac{1}{2} \left(\pi_t(x)^2 + (L^{-1}(t)\partial_x \phi_t)(x)^2 \right)$$
$$= \frac{1}{2} \sum_{n=1}^{\infty} \left[p_t(n)^2 + \left(\frac{n\pi}{L(t)} \right)^2 q_t(n)^2 \right] \qquad \text{(III. 18)}$$

One can then proceed to define suitable creation and annihilation operators

$$a_t(n) = \left(\frac{n\pi}{2L(t)} \right)^{1/2} q_t(n) + i \left(\frac{L(t)}{2n\pi} \right)^{1/2} p_t(n) \qquad \text{(III. 19)}$$

thereby making concrete the choice of our representation, and inheriting $\mathcal{E}$ with a particular complex structure [15].

With respect to these, the Hamiltonian is written

$$H_t = \sum_{n=1}^{\infty} \frac{n\pi}{L(t)} a_t^\dagger(n) a_t(n) + E^0(t) \qquad \text{(III. 20)}$$

The terms $E^{vac}(t)$ stands for $\sum_{n=1}^{\infty} \frac{n\pi}{2L(t)}$ that corresponds to vacuum "energy". This is formally divergent and cannot be physically normal ordered away. Indeed with proper regularisation this is exactly the Casimir energy of the field between the plates. In our case it is rather straightforward to calculate it using the standard point - splitting methods [17], with the boundary condition that $E^0 = 0$ for $L(t) \to \infty$ pointwise. Indeed, the t- dependence has no consequence in the computational details. The result is simply

$$E^0_{ren}(t) = -\frac{\pi}{3L(t)} \qquad \text{(III. 21)}$$

In order to avoid a natural misunderstanding , we should stress here that E^0 is *not* the physical vacuum energy of the fields, i.e. it is not the value of energy one would read when, say, measuring the force on the plates. It is rather the lowest *possible* energy the field can have at any time, and is not expected to be ever naturally realised at all times even when the system starts in a vacuum state (except of course for the trivial case that $L(t)$ is constant). The physical energy at any time is to be determined by a proper examining of the energy histories and is expected to be a sum of E^0 with energy due to the "particle creation".

The spectrum of H_t is easily identified from (3.20). The vacuum $|0\rangle$ (identified with the vector $|\exp 0\rangle$) satisfies

$$a_t(n)|0\rangle = 0 \qquad \text{(III. 22)}$$

11

and corresponding "many particle" states

$$|n_1, t_1; \ldots; n_s t_s\rangle = a_{t_1}^\dagger(n_1) \ldots a_{t_s}^\dagger(n_s)|0\rangle \qquad \text{(III. 23)}$$

interpreted as corresponding to the proposition that one quantum characterised by n_i is present at time t_i for $i = 1 \ldots s$. The corresponding energy is given by the eigenvalues, e.g.

$$H_t|t_1, n_1\rangle = \delta(t - t_1)\left(\frac{n_1\pi}{L(t)} + E^0(t)\right)|t_1, n_1\rangle \qquad \text{(III. 24)}$$

To be precise the Hamiltonian, the annihilation and creation operators as well as the eigenvalues can be rigorously defined only with respect to smearing. The smearing functions for $a\ a^\dagger$ and the eigenstates are elements of $\mathcal{E}$, or since n is a discrete quantity with elements of $L^2(R)$, e.g. can meaningfully define

$$a_n(z) = \int dt z(t) a_t(n) \qquad \text{(III. 25)}$$

$$|n, \psi\rangle = \int \psi(t)|n, t\rangle \qquad \text{(III. 26)}$$

The smeared Hamiltonian $H_\xi = \int dt \xi(t) H_t$ is a well defined operator on $\mathcal{F}$. This can be seen, in direct analogy with [9] by considering the automorphism

$$e^{-iH_\xi} a_t(n) e^{iH_\xi} = e^{-i\frac{n\pi}{L(t)}\xi(t)} a_t(n) \qquad \text{(III. 27)}$$

which implies that H_ξ can be rigorously defined on $\mathcal{F}$ by its action on coherent states. If we denote by $\Gamma(A)$ the operator on $\mathcal{F} = \exp \mathcal{E}$ defined as

$$\Gamma(A)|\exp z(.)\rangle = |\exp Az(.)\rangle \qquad \text{(III. 28)}$$

where A an operator on $\mathcal{E}$. Also defining $d\Gamma(A)$ by

$$\Gamma(e^A) = e^{d\Gamma(A)} \qquad \text{(III. 29)}$$

we can verify that

$$H_\xi = id\Gamma(-ih_\xi) \qquad \text{(III. 30)}$$

with h_ξ acting on $\mathcal{E}$ by

$$(h_\xi z)_n(t) = \frac{n\pi}{L(t)}\xi(t) z_n(t) \qquad \text{(III. 31)}$$

Equation (3.30) implies that H_ξ is actually definable for all measurable $\xi(t)$ [9].

III.4 The decoherence functional

We have now all the necessary information to compute the decoherence functional:

III.4.1 The Heisenberg operator

To construct the Heisenberg operator we start from equation (2.14) and have to take into account the time-dependence of the Hamiltonian and the continuity of time.

Let us first assume that time t takes values in the interval (t_0, t_f). This can be taken as $(-\infty, \infty)$ if one wishes, but then square integrability forces that the boundary Hilbert spaces are just $\mathbf{C}$ and contain only the vacuum state.

Now, concerning the time - dependence it is straightforward to verify that each operator U_t at a single time Hilbert space $\mathcal{H}_t$ ought be implemented by

$$U_t|\exp w\rangle_{\mathcal{H}_t} = |\exp \hat{u}_t w\rangle_{\mathcal{H}_t} \tag{III. 32}$$

with $\hat{u}_t$ a unitary operator acting on $\mathcal{H}_t$ by

$$(\hat{u}_t w)_n = \exp\left(-in\pi \int_{t_0}^{t} ds/L(s)\right) w_n \tag{III. 33}$$

Let us now pass to the continuous case. Using equation (3.15) we have

$$\mathcal{U}|\exp w(.)\rangle_{\mathcal{F}} = \otimes_t U_t|\exp w_t\rangle_{\mathcal{H}_t}$$
$$= \otimes_t|\exp \hat{u}_t w_t\rangle_{\mathcal{H}_t} = |\exp U w(.)\rangle_{\mathcal{F}} \tag{III. 34}$$

where

$$(Uw)_n(t) = \exp\left(-in\pi \int_{t_0}^{t} ds/L(s)\right) w_n(t) \tag{III. 35}$$

Hence

$$\mathcal{U} = \Gamma(U) \tag{III. 36}$$

III.4.2 The Schrödinger operator

In the construction of the decoherence functional the operator $\mathcal{S}$ contains the information of the natural way the single-time Hilbert spaces are welded together. It involves a natural isomorphism between different - time Hilbert spaces. A natural expression for such an isomorphism would be to consider the map

$$I(t+a, t)|\exp w\rangle_{\mathcal{H}_t} = |\exp w\rangle_{\mathcal{H}_{t+a}} \tag{III. 37}$$

which is essentially the unitary operator connecting b_t with b_{t+a} [4]. But this would fail to perform the transformation $\phi_t \rightarrow \phi_{t+a}$, which is deemed essential, if this would be to connect the representations of the canonical groups at different

---

[4]Note that in this the index t is simply labeling the Hilbert space, b_t meaning an annihilation operator in the single time Hilbert space $\mathcal{H}_t$. We are using the letter b to distinguish from the operator valued distribution a_t defined previously as acting on $\mathcal{F}$.

single time Hilbert spaces. The identification of $\mathcal{H}_t$ at different t ought to be given in terms of the physical variables of the theory, rather than the annihilation and creation operators, related to the non - invariant notion of particle.

This being our concrete physical principle for the identification of time translations, we can easily establish that these ought to be implemented on the b and $b^\dagger$ as

$$b_t(n) \rightarrow \frac{1}{2}\left(\sqrt{\frac{L(t+a)}{L(t)}} + \sqrt{\frac{L(t)}{L(t+a)}}\right)b_{t+a}(n)$$
$$+ \frac{1}{2}\left(\sqrt{\frac{L(t+a)}{L(t)}} - \sqrt{\frac{L(t)}{L(t+a)}}\right)b_{t+a}^\dagger(n) \qquad \text{(III. 38)}$$

But can this transformation be unitarily implemented? Before addressing this, let us recall a number of useful objects appearing in the Fock space construction. Let V_t^* denote the complex conjugate of the Hilbert space V_t [5], with the act of complex conjugation defining an anti - linear isomorphism $C : V_t \rightarrow V_t^*$. Clearly $C^{-1} = -C^\dagger$.

An element $X(w)$ of the Lie algebra of the canonical group at time t is parametrised by elements w of V_t as $X_t(w) = b(Cw) + b^\dagger(w)$. In terms of its matrix elements we have

$$\langle \exp z|X_t(w)|\exp\phi\rangle = [(w,\phi)_{V_t} + (z,w)_{V_t}]\,\langle \exp z|\exp\phi\rangle \qquad \text{(III. 39)}$$

With this notation, the unitary operator $I(t+a,t)$ (which we shall denote I_a) ought to act

$$I_a^\dagger\left(b_t(Cw) + b_t^\dagger(w)\right)I_a = b_{t+a}(CA_aw + B_aw) + b_{t+a}^\dagger(A_aw + C^{-1}B_aw) \quad \text{(III. 40)}$$

in terms of the two operators

$$A_a : V_t \quad \rightarrow \quad V_{t+a}$$
$$(A_aw)_n \quad = \quad \frac{1}{2}\left(\sqrt{\frac{L(t+a)}{L(t)}} + \sqrt{\frac{L(t)}{L(t+a)}}\right)w_n \qquad \text{(III. 41)}$$
$$B_a : V_t \quad \rightarrow \quad V_{t+a}^*$$
$$(B_aw)_n \quad = \quad \frac{1}{2}\left(\sqrt{\frac{L(t+a)}{L(t)}} - \sqrt{\frac{L(t)}{L(t+a)}}\right)w_n \qquad \text{(III. 42)}$$

These operators can be easily checked to satisfy the Bogolubov identities identities

$$A_a^\dagger A_a - B_a^\dagger B_a = 1 \qquad \text{(III. 43)}$$
$$A_a^\dagger \bar{B}_a = B_a^\dagger \bar{A}_a \qquad \text{(III. 44)}$$

---

[5]Recall that V_t by virtue of the representation of the history group can be viewed as a real vector space with a specific complex structure.

where we have denoted $\bar{B} = C^{-1}B$. Also we shall denote $\bar{A} = CAC^{-1}$.

Unfortunately I_a turns out not to be unitarily implementable. The necessary condition for this would

$$Tr(K_a^\dagger K_a) < \infty \qquad \text{(III. 45)}$$

where $K_a : V_t \to V_{t+a}^*$ is defined by $K_a = \bar{B}_a \bar{A}_a^{-1}$ In our case $K_a^\dagger K_a$ is

$$(K^\dagger K w)_n = \left(\frac{L(t+a) - L(t)}{L(t+a) + L(t)} \right)^2 w_n \qquad \text{(III. 46)}$$

This is proportional to unity and hence its trace diverges.

This is not surprising, it is the reason for the well-known inability to define an instantaneous Hamiltonian in any field theory in non-static spacetimes[6]. But this problem is not as serious in a histories theory as in a canonical scheme. The intertwiners between the different representations of the canonical group may not exist, but we should recall that the physical object in any history theory is the decoherence functional. The presence of continuous time (the important difference between this scheme and standard quantum mechanics) implies a smearing in time of all physical quantities and will turn out to be crucial in our ability to provide a well defined decoherence functional.

Let us start by regularising the operator I: we shall consider an ultraviolet integer cut-off N in the field modes. The regularised operator $I^{(N)}$ is then well defined, and can be readily found to have the matrix elements

$$_{\mathcal{H}_{t+a}}\langle \exp w | I_a^{(N)} | \exp z \rangle_{\mathcal{H}_t} = [\det(1 - \bar{K}_a K_a)]^{-1/2}$$
$$\exp\left(-\frac{1}{2}(w, K_a^{(N)} w) - \frac{1}{2}(\bar{K}_a^{(N)} z, z) + (A^{-1} w, z) \right) \qquad \text{(III. 47)}$$

At the limit $N \to \infty$ the Fredholm determinant diverges. The important point, for what follows, is that when a is taken to be infinitesimal, that is, equal to δt the N-dependence appears solely in the terms of order $(\delta t)^2$, and higher. It is easy to check that for small small δt, K_a is proportional to δt and since in the divergent determinant K_a appears squared the lowest divergent contribution is of order $(\delta t)^2$. This point is of primary importance for the construction of a cut-off independent decoherence functional and as a mathematical fact it is not restricted to our particular model.

Let us now begin our construction of the Schrödinger operator. We shall start by considering its discrete version using the regularised intertwiner. Let

---

[6] We should point out that time translation does not constitute in an sense a symmetry of the system (See Ref. [13] for the implementation of symmetries in histories theories.). This also implies that the action operator introduced in [10] as the generator of physical time transformations does not exist. This is a consequence of the fact that such systems as quantum field theory in non - static backgrounds are effectively "open". The full theory ought to include the interaction with gravitational field, in which case (as it is true at the classical level) the Schrödinger time could be considered as homogeneous. In this case the action operator might be expected to exist.

us assume then a discrete n-time history with propositions in the interval $[t_i, t_f]$ such that $t_i = t_0$ and $t_f = t_n$. Let the time interval between any two propositions be constant and equal to $\delta t = (t_f - t_i)/n$. We have

$$\langle \exp w_{t_0}; w_{t_1}; \ldots; w_{t_n} | \mathcal{S} | \exp z_{t_0}; z_{t_1}; \ldots z_{t_n} \rangle$$

$$= \langle \exp w_{t_0} | I_{\delta t}^{(N)} | \exp z_1 \rangle \langle \exp w_{t_1} | I_{\delta t}^{(N)} | \exp z_2 \rangle \ldots \langle \exp w_n | I_{t_n - t_0}^{(N)} | \exp z_1 \rangle$$

$$= \langle \exp w_{t_f} | I_{t_n - t_0}^{(N)} | \exp z(t_0) \rangle \exp \left(\sum_{k=0}^{n-1} ((w_k, z_{k+1}) \right.$$

$$\left. - \frac{\dot{L}(t_k)}{4L(t_k)} ((w_{t_k}, C w_{t_k}) + (C^\dagger z_{t_{k+1}}, z_{t_{k+1}})) \delta t \right) + O[(\delta t)^2] \right) \text{(III. 48)}$$

Now at the limit of $\delta t \rightarrow \infty$ the term in the exponential in (3.48) becomes a Stieljes integral: hence at the continuous limit one can meaningfully write

$$\langle \exp w(.) | \mathcal{S} | \exp z(.) \rangle = \langle \exp w(t_f) | I_T^{(N)} | \exp z(t_0) \rangle$$

$$\times \exp \left(\sum_{n=1}^{\infty} \left(w_n^*(t_f) z_n(t_f) + \int_{t_0}^{t_f} ds (w_n^*(s) z_n(s) + w_n^*(s) \dot{z}_n(s) \right. \right.$$

$$\left. \left. - \frac{\dot{L}}{4L}(s) [w_n^*(s) w_n^*(s) + z_n(s) z_n(s)] \right) \right)$$

$$:= \langle \exp w(t_f) | I_T^{(N)} | \exp z(t_0) \rangle e^{iA[w(.), z(.)]} \text{(III. 49)}$$

Hence the only divergent conribution to $\mathcal{S}$ comes from the operator $I^{(N)\dagger}$ which appears in the boundary term. But it is the boundary term that is multiplied by the boundary operator $\mathcal{A}$. To see how this multiplication is to be carried out let us return to equations (2.10) and (2.12). The indices i and j in equation (2.10) correspond respectively to the initial and final Hilbert space. Hence, when due to equation (2.12) the density matrix at the opposite boundary acts upon them one should introduce a factor of $I_T^{(N)}$. Hence if one writes $\mathcal{A}$ in a coherent state basis on $\mathcal{H}_{t_0} \otimes \bar{\mathcal{H}}_{t_n}$ it should read

$$\langle \exp w_f | A^{u_f v_0} | \exp z_0 \rangle = \langle \exp w_f | (\rho_f)^{1/2} | \exp u_f \rangle \langle \exp v_0 | (\rho_0)^{1/2} I_T^{\dagger(N)} | \exp z_f \rangle$$
$$\times [Tr(I_T^{(N)} \rho_f I_T^{\dagger(N)} \rho_0)]^{-1/2} \qquad \text{(III. 50)}$$

where the presence of the $I^{(N)}$ in the denominator is to make the trace well defined and $T = t_n - t_0$. Hence we can compute the map $\mathcal{A}\mathcal{S}$ (properly speaking, the multiplication is to be performed in the discrete history version, but the e^{iA} term is not affected anyway). Hence our result reads

$$\langle \exp w(.) | \mathcal{S} \mathcal{A}^{u_f v_0} | \exp z(.) \rangle = \langle \exp w(t_f) | (\rho_f)^{1/2} | \exp u_f \rangle \langle \exp v_0 | (\rho_0)^{1/2} | \exp z(t_0) \rangle$$
$$\times e^{iA[w(.), z(.)]} [Tr(I_T^{(N)} \rho_f I_T^{\dagger(N)} \rho_0)]^{-1/2} \qquad \text{(III. 51)}$$

16

There is still a N-dependence in the denominator, *but this cancels out* when we restrict ourselves to the standard case of time-asymmetric histories with $\rho_f = 1$.

Hence eventually, we have arrived at a well-defined, finite expression for the decoherence functional for the time-asymmetric case. It is of the form (2.19) with the Heisenberg operator given by (3.36) and the map $\mathcal{S}\mathcal{A} : \mathcal{F} \rightarrow \mathcal{H}_{t_i} \otimes \bar{\mathcal{H}}_{t_f}$ given by the matrix elements (3.51).

For completeness we give the following expression involving the trace over $\mathcal{H}_{t_0} \otimes \bar{\mathcal{H}}_{t_f}$

$$Tr_{\mathcal{H}_{t_0} \otimes \bar{\mathcal{H}}_{t_f}} (\mathcal{S}\mathcal{A})^\dagger (\mathcal{A}\mathcal{S}) : \mathcal{F} \otimes \bar{\mathcal{F}} \otimes \mathcal{F} \otimes \bar{\mathcal{F}} \rightarrow \mathbf{C}$$

$$|\exp w(.)\rangle \otimes \langle \exp z(.)| \otimes |\exp w'(.)\rangle \otimes \langle \exp z'(.)| \rightarrow$$

$$\langle \exp w(t_f)| \exp z'(t_f)\rangle \langle \exp w'(t_0)|\rho_0| \exp z(t_0)\rangle e^{iA[w(.),z(.)] - iA^*[w'(.),z'(.)]} \quad \text{(III. 52)}$$

Let us at this point address an important mathematical subtlety. The Gaussian integral over the coherent states is essentially an integral of the Wiener type defined primarily on skeleton paths (cylinder sets) and then by continuity extending to the whole of the Hilbert space. As such one should be very careful when taking the continuum limit for the path.

This is in particular important for equation (3.49). There we have written a term $\int dt w^*(t)(1 + \frac{\partial}{\partial t})z(t)$ as a limit of the discretised term $\sum_k w_k^* z_{k+1}$. It should be kept in mind that this is just a formal suggestive expression, so should not be taken literally. This should be more correctly be written as

$$\int dt w^*(t)(1 + \partial_-)w(t) \quad \text{(III. 53)}$$

Where ∂_- is the *backwards* in time Ito derivative defined by the limit of the following matrix in skeleton paths

$$\begin{pmatrix} ... & ... & ... & ... \\ 1 & -1 & 0 & ... \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad \text{(III. 54)}$$

Also note that in $\mathcal{S}^\dagger$ the corresponding term is $1 - \partial_+$ with $\partial_+ = -\partial_-^\dagger$. For a large class of calculations this distinction might not be proved important, but whenever one encounters determinants and inverses, one should regularise and then take the continuum limit. This is the case for the calculations performed in the Appendix. For more details see for instance the reference [18].

III.5   Some examples

From our previous results it is easy to write down an explicit expression for the decoherence functional evaluated for particular choices of history propositions.

III.5.1   Coherent state histories and the path integral

The projection operators

$$P_{w(.)} = |\exp w(.)\rangle\langle\exp w(.)|/\langle\exp w|\exp w\rangle \qquad \text{(III. 55)}$$

corresponds to propositions about coherent state paths [8]. It is straightforward to compute that

$$d(w(.),z(.)) = \langle\exp w'(t_0)|\rho_0|\exp z(t_0)\rangle\langle\exp w'(t_f)|\exp z'(t_f)\rangle$$
$$\times e^{iS[w(.)]-iS^*[z(.)]} \qquad \text{(III. 56)}$$

with S the coherent state action

$$iS[w(.)] = \sum_{n=1}^{\infty}\left(w_n^*w_n(t_f) - \int_{t_0}^{t_f} ds \left(w_n^*(s)\dot{w}_n(s) - i\frac{n\pi}{L(s)}w_n^*(s)w_n(s)\right.\right. \qquad \text{(III. 57)}$$

$$\left.\left. +\frac{\dot{L}}{4L}(s)[w_n^*(s)w_n^*(s) + w_n(s)w_n(s)]\right)\right) \qquad \text{(III. 58)}$$

The above formula has such a strong similarity to a coherent state path integral expression for the decoherence functional, that we cannot help wonder whether such an object is meaningful.

In some (not very precise) sense it is. If one wants to evaluate the time-evolution kernel $_{\mathcal{H}_{t_f}}\langle w_f;t_f|w_i;t_i\rangle_{\mathcal{H}_{t_0}}$ one can proceed by the standard way by splitting the interval $[t_0,t_f]$ into intervals of width δt and considering evolution first by the intertwiner $I_{\delta t}$ and then by Hamiltonian evolution. It would be then easy to repeat the derivation that at the limit $\delta t \to 0$ the amplitude becomes N - independent, reading

$$_{\mathcal{H}_{t_f}}\langle w_f;t_f|w_i;t_i\rangle_{\mathcal{H}_{t_0}} = \int DwDw^* e^{iS[w(.),w^*(.)]} \qquad \text{(III. 59)}$$

with summation such that $w(t_0) = w_i$ and $w^*(t_f) = w_f^*$. But of course this would be just a formal expression since ne cannot interchange the limit of $\delta t \to 0$ with the integrations. In addition, it would be rather awkward to have the standard quantum theory with changing Hilbert spaces. This construction is natural only in a history framework.

Still, it would be interesting to look for a rigorous definition of the path integral (3.57). I am in particular referring to Klauder's algorithm [19] of constructing the coherent state path integral through the use of a metric on phase space, so that one can define a Wiener process upon it. It is a conjecture, worth investigating, that such an object could be constructed from the introduction of a time - dependent metric on phase space. The natural candidate would be the standard: the pullback of the projective Hilbert space metric to phase space

$$ds^2(t) = ||\mathbf{d}|qp;t\rangle||^2 - |\langle qp;t|\mathbf{d}|qp;t\rangle|^2$$

$$= \sum_{n=0}^{\infty} \left(\frac{n\pi}{2L(t)} dq_n^2 + \frac{L(t)}{2n\pi} dp_n^2 \right) \qquad \text{(III. 60)}$$

where $\mathbf{d}$ is the exterior differentiation operator on phase space.

III.5.2 Particle creation

The best way to examine the effect of particle creation is through the consideration of energy histories. That is we need to consider the value of the decoherence functional on coarse grained projectors in values of energy.

The linearity of the field allows us to separately consider the effect on each mode. Restricting ourselves to any mode labeled by n we can easily verive that the most general projector onto energy eigenstattes is of the form

$$P = \sum_{r=0}^{\infty} \int dt_1 \ldots dt_n \kappa_r(t_1, \ldots, t_r) |t_1 \ldots t_r\rangle\langle t_1 \ldots t_r| \qquad \text{(III. 61)}$$

where κ_r are step functions elements of $L^2(R^r)$ and correspond to smearing with respect to time of a proposition about r quanta. Substituting these into equation (2.19) we could easily get an expression for the decoherence functional. Of particular interest is of the case where

$$\kappa_r(t_1, \ldots, t_r) = \chi_\Delta(t_1) \ldots \chi_\Delta(t_m) \delta_{rm} \qquad \text{(III. 62)}$$

correponding to a proposition of a appearance of m quanta (of the quantum number n) within the time interval Δ (χ_Δ stands for the characteristic function of this interval).

It would be indeed cumbersome to conduct the full analysis of finding the proper coarse graining of the energy histories that would allow us to identify consistent sets of energy histories and hence of the quasiclassical values of the total energy in the cavity.

Still, we can make a number of qualitative statements solely through the analysis of the logic of this construction. Essentially, the continuous history approach has enabled us to define a quantum theory by considering propositions of quantities smeared in time. Hence a proposition about energy is meaningful not when defined sharply at a moment, but rather as a proposition about the time - averaged number of excitations in a time interval of width Δt. Hence when evaluating the time average energy in an interval Δt one has an effective high energy cut -off at the mode number

$$N \simeq \frac{L}{\Delta t} \qquad \text{(III. 63)}$$

From equation (3.42) one can estimate that (at the classical limit) the total number of particles of mode n created in an interval Δt is of the order of

$(\dot{L}\Delta t/L)^2$, so that the average energy should peak around the value (ignore constants of order one)

$$\bar{E} \simeq \sum_{n=1}^{N} \left(\frac{\dot{L}\Delta t}{L} \right)^2 \frac{n}{L} \simeq \frac{\dot{L}^2 (\Delta t)^2}{L^3} N^2 \simeq \frac{\dot{L}^2}{L} \qquad \text{(III. 64)}$$

We can also estimate an optimal degree of coarse graining that minimises the energy uncertainy ΔE. This ought to have two contributions , the fully quantum (proportional to $(\Delta t)^{-1}$) and a statistical one associated with time averaging. The latter will be essentially

$$(\Delta E)_{av} \simeq \frac{\dot{L}^2 (\Delta t)^2}{L^3} N \simeq \frac{\dot{L}^2 \Delta t}{L^2} \qquad \text{(III. 65)}$$

hence the total energy fluctuation will behave as

$$\Delta E \simeq (\Delta t)^{-1} + \frac{\dot{L}^2 \Delta t}{L^2} \qquad \text{(III. 66)}$$

which is minimised at $\Delta t \simeq \frac{L}{\dot{L}}$ Hence $\Delta E = \frac{\dot{L}}{L}$ is a minimum degree of energy coarse graining that will possibly lead to consistency of time - averaged energy histories . Since $\Delta E >> \bar{E}$ the classical picture we would get would be of large *classical* fluctuations around the minimum value $E^0(t)$. The fluctuations should be adequately described by a noise term of amplitude $\dot{L}/L$ when sampling in times larger than $L/\dot{L}$.

III.5.3   CTP generating functional and n-point functions

The decoherence functional written in (2.19) is defined on pairs of projectors of the history Hilbert space. As such it can be extended by continuity to act on all bounded operators there. A particular instance is of course the smeared field operators. It can be readily checked that if for $P_{\alpha'}$ we substitute $\phi(X)\phi(Y)$ and for P_α the unity in (2.19), the value of the decoherence functional is nothing but the Feynman propagator for the field, i.e. the expectation value of the time - ordered product of two fields.

This is an important point, because this means that from the decoherence functional one can read objects appearing in the standard canonical quantum mechanical treatment. More generally if for $P_{\alpha'}$ we substitute the product $\phi(X_1)\ldots\phi(X_n)$ and for P_α $\phi(Y_1)\ldots\phi(Y_m)$ (smeared of course with suitable test functions) we obtain in an obvious notation

$$d(\phi(Y_1)\ldots\phi(Y_m),\phi(X_1)\ldots\phi(X_n)) = G^{(n,m)}(X_1,\ldots X_n, Y_1, \ldots Y_m) \quad \text{(III. 67)}$$

where $G^{(n,m)}$ are the Schwinger - Keldysh close- time - path (CTP) correlation functions : for $m = 0$ they are the time - ordered and for $n = 0$ the anti -time

ordered correlation functions. The corresponding generating functional is then readily defined as

$$Z_{CTP}[J_+, J_-] = d(e^{-i\hat{\phi}(J_-)}, e^{i\hat{\phi}(J_+)}) \qquad \text{(III. 68)}$$

in terms of the smearing functions $J_+(t, x)$ and $J_-(t, x)$ interpreted as external sources. Or still we can generalise to a phase space closed - time -path (PSCTP) generating functional

$$Z_{PSCTP}[(f, g)_+, (f, g)_-] = d(U^\dagger(f_-, g_-), U(f_+, g_+)) \qquad \text{(III. 69)}$$

in terms of the generators $U(f, g)$ of the history group [7].

These equations can be easily verified to give the correct results in the static case $\dot{L} = 0$. They also provide well defined and finite objects in the general time - dependent case. This means in particular that the $n - point$ functions of this theory are meaningful distributions. Again we should stress the importance of smearing over time. In the canonical quantisation scheme, time is not treated in the same footing as the spatial variables, it is not a variable with respect to which one actually smears. The n-pt functions are then strictly speaking smooth functions of their t - arguments. Here, smearing over t implies that the n-point functions are of non - trivial distributional character with respect to all their variables.

Let us consider the easiest case, where time runs in the full real axis and the initial state is the vacuum. Then the time - ordered two - point function is (recall $Tr\mathcal{S} = 1$).

$$G^{(2,0)}(t, x; t', x') = Tr\left(\mathcal{U}\phi_t(x)\phi_{t'}(x')\mathcal{U}^\dagger\mathcal{S}^\dagger\right) \qquad \text{(III. 70)}$$

and can actually be computed (see the Appendix) as

$$G^{(2,0)}(t, x; t', x') = \sum_n \frac{L(t) + L(t')}{[L(t)L(t')]^{1/2}} \frac{1}{4n\pi} \sin(n\pi x) \sin(n\pi x') \left[e^{-in\pi|F(t)-F(t')|}\right]$$
$$+ \frac{1}{L(t)}\delta(t - t')\delta(x - x') \qquad \text{(III. 71)}$$

where $F(t) = \int_{-\infty}^t ds/L(s)$. This is to be compared with the expression used by Davies and Fulling [22].

We should note here that in a histories framewok there is no direct physical interpretation for the n -point functions. It is difficult to view them as expectation values of time - ordered products of the fields, simply because we need

---

[7]The relation of the generating functional to the decoherence functional is reminiscent of the definition of the generating functional of a stochastic process as a Fourier transform of the stochastic probability measure. In the quantum case we have a double Fourier transform which reflects the fact that the decoherence functional represents a "quantum measure" on phase space [20, 21].

no assume an ensemble of systems - hence expectation value is a vague or even meaningless concept. This of course relates to the old problem of what meaning quantum mechanical probability has when talking about a single system. In any case, the n -point functions arise naturally as temporal correlation functions from which the CTP generating functional (and consequently the decoherence functional) is constructed. Indeed we would expect an analogue of Wightman 's reconstruction theorem [16] to hold in the histories version of quantum field theory. By that we mean that the knowledge of the hierarchy of all CTP correlation functions (satisfying certain spacetime symmetry requirements) should uniquely determine the histories Hilbert space $\mathcal{F}$, the decoherence functional and a representation of the group of spacetime symmetries on $\mathcal{F}$.

We should also remark that the study of the short distance behaviour of the n - point functions will enable us to determine whether they are of the Hadamard form [15, 23] (we do expect that for vacuum initial states taken as in the example we calculated above, but not for generic initial conditions). This will enable a direct comparison between the histories quantisation and the C^* - algebraic framework [8] for the description of quantum fields in curved spacetime (see [15] and references therein).

IV   Conclusions

We have seen the construction of a well defined, finite quantum theory describing an 1+1 field in a time dependent cavity . This has been written in a continuous - time histories form, using recently developed ideas and techniques and in this context being based on the used of smeared in time observables to ensure finiteness of our objects. It is in this sense important that we have been able to *a posteriori* justify our construction by relating it to the CTP formalism .

We should remind again the reader the two important principles - one mathematical one physical - entering our construction. The first is the use of the history group and the requirement of existence of an instantaneous Hamiltonian as posited in [8, 9]. The second is the appearance of two distinct notions of time - transformations as identified in [10]. Their separation ought to be reflected in the probability assignment, hence in the decoherence functional. In a concrete sense, our result points implies the construction of quantum theories in a histories scheme, that cannot be satisfactorily defined in a canonical way [9].

From the perspective of the current paper the next step will be to apply these principles to provide a *unique* algorithm for constructing a generic quantum field theory in curved spacetime. Our approach does straightforwardly apply in this case (for spacetimes with compact Cauchy surfaces and in the absence of zero

---

[8]In particular the GNS construction of a Hilbert space where the C^* - algebra is represented based on the choice of Hadamard vacua.

[9]Another example, this time in classical systems, where constructions using histories yield better results than the corresponding canonical ones is to be found in [24].

modes); the important unaddressed issue is whether the resulting history theory is dependent or not in the choice of the time variable: essentially whether the linear transformations associated with the change of foliation can be implemented by *proper* Bogolubov transformations. This is a difficult problem both at the technical and conceptual level and we hope to address it in a future work.

We should point out, that quantisation based on the history group has a number of advantages over its canonical counterpart:

1. It has a much smaller degree of arbitrariness in the choice of the representation. If, as we believe, changes of foliation turn out to be unitarily implementable , then there will be no ambiguity at all in the quantisation algorithm.
2. It allows a real - time description of field observables, rather than focusing on evaluating the S - matrix between in and out vacua. In particular, global quantities (such as time - averaged total energy) can be unambiguously identified and we are allowed to make predictions or assign probabilities about their values [10].
3. As a quantisation algorithm it is strictly local. That means, if we restrict ourselves in a thin spacetime slice the theory is constructed from the knowledge of the causal structure ($\mathcal{S}$), the dynamics ($\mathcal{U}$) and the initial data ($\mathcal{A}$). While the canonical approach necessitates a choice of positive frequency solutions and most physical criteria for such a choice necessitate a knowledge of the behaviour of classical solutions at all times.

If our construction passes the test of unitarily incorporating changes of foliation, this would be an impetus for further generalisation. Eventually, we would like to treat fields in spacetimes that are not globally hyperbolic: ideally the ones appearing in the black hole evaporation process [26].

V Aknowledgements

---

[10] Even at the current state of the theory, where the general covariance of the scheme remains unproven, our results should be relevant to the study of cosmological models, where a preferred foliation is always assumed. The strength of this approach lies in the possibility of providing a unique choice for the correlation functions of the field and as such should facilitate treatment of issues like matter backreaction to the geometry, especially in approaches that employ the CTP formalism (for a sampling of recent work see [25]).

A   Computing the two - point function

From equation (3.67) and the definitions (3.19) we see that the Fourier transform of the time - ordered two - point function $G_{nm}^{(2)}(t, t')$ is by

$$G_{nm}^{(2)}(t, t') = \delta_{nm} G_n(t, t') \tag{A. 1}$$

with

$$G_n(t, t') = \frac{[(L(t)L(t')]^{1/2}}{2n\pi} Tr \left(\mathcal{U}(a_{tn}a_{t'n} + a_{tn}^{\dagger}a_{t'n}^{\dagger} + a_{tn}a_{t'n}^{\dagger} + a_{tn}^{\dagger}a_{t'n}) \mathcal{U}^{\dagger}\mathcal{S}^{\dagger} \right) \tag{A. 2}$$

From (3.35) we can see that

$$\mathcal{U}b_{nt}^{\dagger}b_{t'n}\mathcal{U}^{\dagger} = e^{-in\pi[F(t)-F(t')]} \tag{A. 3}$$

with $F(t) = \int_{-\infty}^{t} ds/L(s)$. Similar expressions hold for the other terms in (A.2). Hence we are left to the calculation of objects of the form $\mathcal{S}^{\dagger}b_t b_{t'}$ etc. These are best computed by differentiation of the generating functional

$$A[F, F^*] = \int DwDw^* \exp \left(\int ds[-w^*(s)\partial_+ w(s) \right.$$

$$\left. + \frac{\dot{L}}{4L}(s)(w^*(s)w^*(s) + w(s)w(s)) + F^*(s)w(s) + w^*(s)F(s)] \right) \tag{A. 4}$$

Let us compute first the case $\dot{L} = 0$. In that case it is a standard result that

$$A[F, F^*] = \exp(\int dsds' F^*(s)\Theta_+(s, s')F(s')) \tag{A. 5}$$

Here Θ_+ is the inverse of ∂_+ and has matrix elements

$$\Theta_+^{-1}(t, t') = \theta(t - t') \tag{A. 6}$$

Note the importance of having identified the operator as ∂_+. For ∂_- we would have

$$\Theta_-^{-1}(t, t') = -\theta(t' - t) \tag{A. 7}$$

In the general case where $\dot{L} \neq 0$ the integral is performed by splitting w into its real and imaginary parts. That is we define

$$x(t) = (\frac{a}{2L(t)})^{1/2}(w + w^*) \tag{A. 8}$$

$$y(t) = i(\frac{L(t)}{a})^{1/2}(w^* - w) \tag{A. 9}$$

24

in terms of some arbitrary positive real number a. Then the integral becomes of the form

$$\int DxDy \exp\left(-\frac{1}{2}X^T A Y + K^T X\right) = \exp\left(K^T A^{-1} K\right) \qquad \text{(A. 10)}$$

with

$$A = \begin{pmatrix} \frac{a}{L(t)}\partial_+ & -i\partial_+ \\ i\partial_+ & \frac{L(t)}{a}\partial_+ \end{pmatrix} \qquad \text{(A. 11)}$$

$$K = \begin{pmatrix} (\frac{a}{2L(t)})^{1/2}(F + F^*) \\ i(\frac{L(t)}{a})^{1/2}(F^* - F) \end{pmatrix} \qquad \text{(A. 12)}$$

The inverse of A is just

$$A^{-1} = \frac{1}{2}\begin{pmatrix} \Theta_+ L/a & i\Theta_+ \\ -i\Theta_+ & a\Theta_+ L^{-1} \end{pmatrix} \qquad \text{(A. 13)}$$

It is therefore easy to compute

$$A[F, F^*] = \exp\left(\frac{1}{2}\int dsds' F^*(s)\left(\sqrt{\frac{L(s)}{L(s')}} + \sqrt{\frac{L(s')}{L(s)}}\right)\theta(s - s')F(s')\right) \qquad \text{(A. 14)}$$

leading to

$$G_n(t, t') = \frac{[(L(t)L(t')]^{1/2}}{4n\pi}\left(\sqrt{\frac{L(t)}{L(t')}} + \sqrt{\frac{L(t')}{L(t)}}\right)e^{-in\pi|F(t)-F(t')|} + \delta(t - t') \qquad \text{(A. 15)}$$

from which (3.68) follows.

References

[1] R. B. Griffiths: *Consistent Histories and the Interpretation of Quantum Mechanics*, J. Stat. Phys. **36**, 219 (1984).

[2] R. Omnès: *Logical Reformulation of Quantum Mechanics: I Foundations*, J. Stat. Phys. **53**, 893 (1988); *The Interpretation of Quantum Mechanics*, (Princeton University Press, Princeton, 1994) .

[3] M. Gell-Mann and J. B. Hartle: *Quantum mechanics in the Light of Quantum Cosmology*, in *Complexity, Entropy and the Physics of Information*, edited by W. Zurek, (*Addison Wesley, Reading (1990)*); *Classical Equations for Quantum Systems*, Phys. Rev. **D47**, 3345 (1993).

[4] J. B. Hartle: *Spacetime Quantum Mechanics and the Quantum Mechanics of Spacetime*, in *Proceedings of the 1992 Les Houches School, Gravitation and Quantizations*, 1993.

[5] C. J. Isham: *Quantum Logic and the Histories Approach to Quantum Theory*, J. Math. Phys. **35**, 2157 (1994); C.J. Isham and N. Linden: *Quantum Temporal Logic and Decoherence Fuctionals in the Histories Approach to Generalized Quantum Theory*, J. Math. Phys. **35**, 5472,(1994).

[6] C. J. Isham, N. Linden and S. Schreckenberg: *The Classification of Decoherence Functionals: an Analogue of Gleason's Theorem*, J. Math. Phys. **35**, 6360 (1994).

[7] C. J. Isham: *Topological and Global Aspects of Quantum Theory*, in *Relativity, Groups and Topology II*, Les Houches 1983.

[8] C. J. Isham and N. Linden: *Continuous Histories and the History Group in Generalized Quantum Theory*, J. Math. Phys. **36**, 5392 (1995).

[9] C. J. Isham, N. Linden, K. Savvidou and S. Schreckenberg: *Continuous Time and Consistent Histories*, J. Math. Phys. **39**, 1818 (1998).

[10] K. Savvidou: *The Action Operator in Continuous - Time Histories*, gr-qc/9811078.

[11] J. S. Schwinger: *Brownian Motion of a Quantum Oscillator*, J. Math. Phys. **2**, 407 (1961); L. V. Keldysh: *Diagram Technique for Nonequilibrium Processes*, Zh. Eksp. Teor. Fiz., **47**, 1515 (1964): for a recent treatment see E. Calzetta and B. L. Hu: *Closed Time Path Functional Formalism in Curved Spacetime: Application to Cosmological Backreaction Problems*, Phys. Rev. **D 35**, 495 (1987).

[12] A. Ashtekar and A. Magnon: *Quantum Fields in Curved Spacetimes*, Proc. Roy. Soc. (London), **A346**, 375 (1975).

[13] S. Schreckenberg: *Symmetry and History Quantum Theory: an Analogue of Wigner's Theorem*, J. Math. Phys. **37**, 6086 (1997);*Symmetries of Decoherence Functionals*, *ibid* **38**, 759 (1997).

[14] F. A. Berezin: *The Method of Second Quantization*, (Academic Press, New York, 1966).

[15] R. Wald: *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*,(University of Chicago Press, Chicago, 1994).

[16] R. F. Streater and A. S. Wightman: *PCT, Spin, Statistics and All That*, (W. A. Benjamin, New York, 1964).

[17] N. B. Birrell and P. C. W. Davies: *Quantum Fields in Curved Space*, (Cambridge University Press, Cambridge,1982).

[18] S. Janson: *Gaussian Hilbert Spaces*, Cambridge University Press, Cambridge (1997).

[19] J. R. Klauder: *Quantization is Geometry, after all*, Ann. Phys. (N.Y.) **188**, 120 (1988).

[20] R. D. Sorkin: *Quantum mechanics as Quantum Measure Theory*, Mod. Phys. Lett. **A9**, 3119 (1994); *Quantum Measure Theory and its Interpretation*, in *Quantum Classical Correspondence*, edited by D.H. Feng and B. L. Hu, (International Press, Cambridge MA, 1997).

[21] C. Anastopoulos: *Selection of Preferred Consistent Sets*, Int. Jour. Theor. Phys. **37**, 2261 (1998).

[22] P. C. W. Davies and S. A. Fulling: *Radiation from Moving Mirrors and from Black Holes*, Proc. R. Soc. Lond. **356**, 237 (1977).

[23] S. A. Fulling, M. Sweeny and R. M. Wald: *Singularity Structure of the Two - Point Function in Quantum Field Theory in Curved Spacetime*, Comm. Math. Phys. **63**, 257 (1978).

[24] I. Kouletsis, *A Classical History Theory: Geometrodynamics and General Field Dynamics Regained*, gr-qc /9801019.

[25] E. Calzetta and B. L. Hu: *Noise and Fluctuations in Semiclassical Gravity*, Phys. Rev. **D49**, 6636 (1994); E. Calzetta, A. Campos and E. Verdaguer: *Stochastic Semiclassical Cosmological Models*, Phys. Rev. **D56**, 2163 (1997) ; R. Martin and E. Verdaguer: *An Effective Stochastic Semiclassical Theory for the Gravitational Field*, gr-qc/9812063.

[26] J. B. Hartle, *Generalized Quantum Theory and Black Hole Evaporation*, gr-qc 980870.